MATH3705 A — Test 4

Name and Student Number:

Total points: 20. No partial marks for Questions 1-3.

Closed book! Non-programmer calculators are allowed!

[**2**] 1. Consider the wave equation $u_{tt} = c^2 u_{xx}$, $0 < x < L, t > 0$, subject to the boundary conditions $u(0,t) = 0, u(L,t) = 0$ and the initial conditions $u(x,0) = f(x), u_t(x,0) = g(x)$. The solution is given by:

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos(\frac{n\pi ct}{L}) + b_n \sin(\frac{n\pi ct}{L}) \right] \sin(\frac{n\pi x}{L}),$$

where:

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Given the wave equation $u_{tt} = 5u_{xx}$, $0 < x < 2$, $t > 0$, subject to the boundary conditions $u(0,t) = u(2,t) = 0$ and the initial conditions $u(x,0) = \sin(2\pi x), u_t(x,0) = 3\sin(4\pi x)$, find a_4.

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

Solution: b.

cf. Part IV, Exercise 13.

[**2**] 2. Find a polynomial solution of the Laplace equation $u_{xx} + u_{yy} = 0$, $0 < x < 1, 0 < y < 1$ with boundary conditions $u(x,0) = x, u(x,1) = 0, u(0,y) = 0, u(1,y) = 1 - y$.

(a) x (b) $x - y$ (c) $1 - y$ (d) $x + y - xy$ (e) $x - xy$

Solution: e.

Note that all f_i are continuous and linear on the boundary. Let $u(x,y) = ax + by + cxy + d$. $u(x,y) = x - xy$.

[2] 3. Consider the following Sturm-Liouville problem $y'' + 3y' + \lambda y = 0, \quad y(0) = y(2) = 0$. The weight function is:

(a) e^{3x} (b) e^{2x} (c) $3x$ (d) $2x$ (e) $\sin(2x)$

Solution: (a).

The integrating factor is $e^{\int 3dx}$.

[7] 4. Consider the Laplace equation $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ outside the circle $r = a$, subject to the boundary condition $u(a, \theta) = f(\theta)$, where f is continuous, 2π-periodic, f' is piecewise continuous. Let $u(r, \theta)$ be bounded outside the circle $r = a$. Then the solution of the PDE above is

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}[a_n \cos(n\theta) + b_n \sin(n\theta)],$$

where

$$a_n = \frac{a^n}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta) d\theta, \quad n \geq 0; \quad b_n = \frac{a^n}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta) d\theta, \quad n \geq 1.$$

Question: Find the bounded solution of $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ outside the circle $r = 2$, subject to the boundary condition $u(2, \theta) = 4\cos^2(\theta) - 3\sin(3\theta)$.

Solution: $a = 2$. the solution of the PDE above is

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}[a_n \cos(n\theta) + b_n \sin(n\theta)].$$

Note that $4\cos^2(\theta) = 2[\cos(2\theta) + 1]$. Thus

$$u(2, \theta) = 2 + 2\cos(2\theta) - 3\sin(3\theta).$$

Therefore

$$2 + 2\cos(2\theta) - 3\sin(3\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} 2^{-n}[a_n \cos(n\theta) + b_n \sin(n\theta)],$$

which implies that

$a_0 = 4, 2 = 2^{-2}a_2, a_n = 0$ for $n \neq 0, 2$; $-3 = 2^{-3}b_3$, $b_n = 0$ for $n \neq 3$. Hence

$a_0 = 4, a_2 = 8, a_n = 0$ for $n \neq 0, 2$; $b_3 = -24$, $b_n = 0$ for $n \neq 3$. Hence

$$u(r, \theta) = 2 + 8r^{-2}\cos(2\theta) - 24r^{-3}\sin(3\theta).$$

[7] 5. The solution of the heat equation $w_t = \alpha^2 w_{xx}$, $0 \le x \le L, t \ge 0$, subject to BCs: $w(0,t) = 0$, $w(L,t) = 0$ and IC: $w(x,0) = f(x)$, is given by

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}, \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

Question: Solve the following initial-boundary-value problem for $U(x,t)$:

$$\begin{aligned} U_t &= U_{xx} && 0 \le x \le 2, t \ge 0 && \text{PDE} \\ U(0,t) = 1 \quad U(2,t) &= 3 && t > 0 && \text{BC} \\ U(x,0) = f(x) &= \begin{cases} x+2, & 0 \le x < 1, \\ x+1, & 1 \le x < 2, \end{cases} && && \text{IC} \end{aligned}$$

Solution: Here $\alpha = 1$, $L = 2$, $A = 1$, $B = 3$. Thus

$$v(x) = x + 1. \qquad (1 point)$$

Let $w(x,t) = U(x,t) - v(x)$. Then

$$\begin{aligned} w_t &= w_{xx} && 0 \le x \le 2, t \ge 0 && \text{PDE} \\ w(0,t) &= w(2,t) = 0 && t > 0 && \text{BC} \\ w(x,0) = f(x) - v(x) &= \begin{cases} 1, & 0 \le x < 1, \\ 0, & 1 \le x < 2, \end{cases} && && \text{IC} \end{aligned}$$

Thus, for $n = 1, 2, 3, \ldots$,

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L [f(x) - v(x)]\sin(\frac{n\pi x}{L})\, dx = \frac{2}{2}\int_0^2 [f(x) - v(x)]\sin(\frac{n\pi x}{2})\, dx \\ &= \int_0^1 \sin(\frac{n\pi x}{2})\, dx = \left[-\frac{2}{n\pi}\cos(\frac{n\pi x}{2})\right]_0^1 \\ &= \frac{2}{n\pi}[1 - \cos(\frac{n\pi}{2})]. \end{aligned}$$

$$U(x,t) = x + 1 + \sum_{n=1}^{\infty} \frac{2}{n\pi}[1 - \cos(\frac{n\pi}{2})] \sin\left(\frac{n\pi x}{2}\right) e^{-\left(\frac{n\pi}{2}\right)^2 t}.$$